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	2. Report Date. July 1991	3. Report Type and Dates Covered. Proceedings

4. Title and Subtitle. A Normal Mode Interpretation of a Range Dependent Parabolic Wave Equation	5. Funding Numbers. Program Element No. 61153N; Project No. 03202; Task No. 020; Accession No. DN257033
6. Author(s). Roger M. Oba	
7. Performing Organization Name(s) and Address(es). Naval Oceanographic and Atmospheric Research Laboratory, Ocean Acoustics and Technology Directorate, Stennis Space Center, MS 39529-5004	8. Performing Organization Report Number. PR 91:044:221
9. Sponsoring/Monitoring Agency Name(s) and Address(es). Naval Oceanographic and Atmospheric Research Laboratory, Basic Research Management Office, Stennis Space Center, MS 39529-5004	10. Sponsoring/Monitoring Agency Report Number. PR 91:044:221
11. Supplementary Notes. IMACS '91	
12a. Distribution/Availability Statement. Approved for public release; distribution is unlimited.	12b. Distribution Code.

13. Abstract (*Maximum 200 words*).

Often it is possible to decompose a wave equation into vertical and horizontal components. It is useful to consider the case when the vertical dependence is completely expressed in a vertical normal mode representation. A parabolic wave equation with range dependence transforms to first order linear differential equations in an infinite dimensional Hilbert space. This permits concrete expression of complicated functions of operators and their calculation. The non-constant coefficient matrix shows range dependent vertical mode interactions and constraints on amplitudes. In this study differences and similarities between various transformed equations are explored. The proposed method also permits comparison to existing Hilbert space analysis of the system when the system is stochastic.

14. Subject Terms. Acoustic waves, elastic waves, seismic waves	15. Number of Pages. 2
	16. Price Code.

17. Security Classification of Report.	18. Security Classification of This Page.	19. Security Classification of Abstract.	20. Limitation of Abstract.
Unclassified	Unclassified	Unclassified	SAR

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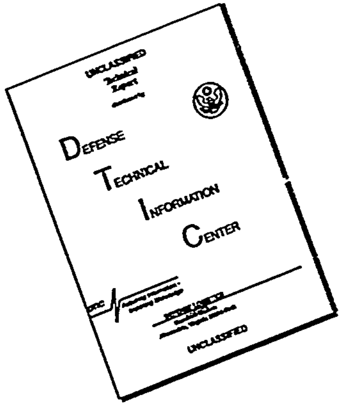

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IMACS '91

13TH WORLD CONGRESS ON COMPUTATION AND APPLIED MATHEMATICS

JULY 22 - 26, 1991
TRINITY COLLEGE DUBLIN
IRELAND

PROCEEDINGS

IN FOUR VOLUMES

VOLUME 1

A NORMAL MODE INTERPRETATION OF A RANGE DEPENDENT PARABOLIC WAVE EQUATION

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ABSTRACT - Often it is possible to decompose a wave equation into vertical and horizontal components. It is useful to consider the case when the vertical dependence is completely expressed in a vertical normal mode representation. A parabolic wave equation with range dependence transforms to first order linear differential equations in an infinite dimensional Hilbert space. This permits concrete expression of complicated functions of operators and their calculation. The non-constant coefficient matrix shows range dependent vertical mode interactions and constraints on amplitudes In this study differences and similarities between various transformed equations are explored. The proposed method also permits comparison to existing Hilbert space analysis of the system when the system is stochastic.

1. INTRODUCTION

Many authors formally derive various parabolic equations in the context of approximating the Helmholtz equation in ocean acoustics (see, *e g.*, McDaniel [1] or Collins [2]). One can, however, use some function analytic methods in Sturm-Liouville theory to make explicit theoretical calculations (Keller and Ahluwalia [3]). McDaniel has used some normal mode theory to discuss various errors of the parabolic approximation [6]. Furthermore, calculations with the generalized Fourier transform give concrete realizations of functions of operators. These allow the statement of certain technical conditions on the range of applicability. Although we start with Tappert's equation [4], the method works for many split step methods and, with certain limitations, the elliptic wave equation. We also restrict our attention to a flat top and bottom surfaces with constant Sturm-Liouville boundary conditions. These approximate pressure release, rigid bottom, or the Pekeris two fluid layer trapped modes (*cf.* Kinsler, Frey, Coppens, and Sanders [5]) This will lead to a completely discrete spectrum, i e , trapped modes only, but the procedure is exactly analogous when there is continuous spectrum corresponding to diffracted modes The latter lead to integral operators rather than matrices, but with the appropriate changes. These differences make for slightly different technical conditions from the discrete case.

2. TRANSFORMING THE PARABOLIC EQUATION

We start with Tappert's original parabolic equation [4]

$$(2ik\partial_r + \partial_z^2 + q)u = 0 \tag{1}$$

where $q = q(r,z) = k^2(n^2-1)$ has r dependence (ignoring approximation to the elliptic Helmholtz equation initially). We write $\partial_z f = f_{,z}$. The boundary conditions will be r independent

$$\cos\theta_\zeta\, u(r,\zeta) - \sin\theta_\zeta\, u_{,z}(r,\zeta) = 0 \tag{2}$$

where $\zeta = 0, -h$ ($\theta_0 = 0$ on a pressure release surface). For a *fixed* r, one may consider the Sturm-Liouville problem in the z variable (see, *e g* , Coddington and Levinson [7] or Weidmann [8])

$$(\partial_z^2 + q)\phi_j + \lambda_j\phi_j = 0 \tag{3}$$

where ϕ_j satisfies the same boundary conditions (2). For continuous, relatively small q one may show the asymptotic behavior

$$\lambda_j \approx h^{-2}(j\pi - \theta_{-h} + \theta_0)^2, \quad j \to \infty \tag{4}$$

Note dependence $\lambda_j = \lambda_j(r)$ and $\phi_j = \phi_j(r,z)$. The independence of the boundary conditions from r means $\phi_{j,r}$ satisfies (2).

Since the $(\phi_j)_{j=1,2,\dots}$ are complete, we may write the generalized Fourier series for u and $\phi_{j,r}$

$$u = \sum b_j\phi_j\ , \quad \phi_{j,r} = \sum a_{jm}\phi_m. \tag{5a,b}$$

The inner product representation for the Fourier coefficient is

$$b_{jm} = b_{jm}(r) = (u,\phi_m) = \int_{-h}^{0} u(r,z)\phi_m dz. \tag{6}$$

Transformation of the parabolic equation requires r derivatives, so differentiate (3), to find

$$[\phi_{j,rzz} + (q+\lambda_j)\phi_{j,r}] + (q_{,r} + \lambda_{j,r})\phi_j = 0 \tag{7}$$

Now use the series for $\phi_{j,m}$ (5b) in the square brackets

$$\sum_j a_{jm}[\phi_{m,zz} + (q+\lambda_j)\phi_m] + (q_{,r} + \lambda_{j,r})\phi_j = 0$$

Rewrite the $\partial_z^2 + q$ in the brackets as an eigenvalue using (3) Finally, the orthonormality of the eigenfunctions, $(\phi_p,\phi_m) = \delta_{pm}$, allows us to pull off individual terms in the series. This results in

$$a_{jp} = -a_{pj} = \begin{cases} 0, & j = p \\ \dfrac{(q_{,r}\phi_j,\phi_p)}{\lambda_p - \lambda_j}, & j \neq p \end{cases} \tag{8}$$

$$\lambda_{p,r} = -(q_{,r}\phi_p,\phi_p). \tag{9}$$

We calculate $0 = (\phi_p,\phi_p)_{,r} = 2(\phi_p,\phi_{p,r}) = 2a_{pp}$. As a special case, suppose the the change in potential $q_{,r}$ is independent of z, *i e.* $q_{,rz} = 0$, then $a_{jp} = 0$ for all j and p, and $\lambda_{p,r} = q_{,r}$ for all p We now combine the previous computations to find the generalized Fourier transformed differential operations on u:

$$(u_{,r},\phi_p) = b_{p,r} + \sum_{j\neq p} a_{pj}b_j \tag{10}$$

in the r direction, and in the z direction

$$([\partial_z^2 + q]u,\phi_p) = -\lambda_p b_p. \tag{11}$$

Define the following semi-infinite matrices

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix},\ A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdot \\ -a_{12} & 0 & a_{23} & \cdots \\ -a_{13} & -a_{23} & 0 & \\ \vdots & & \cdot & \end{bmatrix}, \tag{12}$$

and the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3,\ \ )$ Writing (10) and (11) in matrix form gives the matrix equation for (1)

$$B_r + AB = \tfrac{1}{2ik}\Lambda B. \tag{13}$$

The original differential operators go to the transformed matrix operations as $\partial_r \to \frac{d}{dr} + A$ and $\partial_z^2 + q \to -\Lambda$. We note the following are skew-symmetric ($M^* = \bar{M}^T$ is the adjoint)

$$(\tfrac{1}{2ik}\Lambda - A)^* = -(\tfrac{1}{2ik}\Lambda - A) \tag{14}$$

This leads to the energy conservation statement

$$(u,u) = \sum_j |b_j|^2 = B^*B = \text{constant in } r. \qquad (15)$$

3 SOME CONSEQUENCES

(A) This calculation demonstrates that mode coupling is due entirely to the r dependence of q. The vertical operator $\partial_z^2 + q$ is transformed to the diagonal matrix Λ, which cannot contribute exchange from ϕ_j to ϕ_k, $j \neq k$, via off-diagonal elements. The r derivative, however, transforms to an r derivative plus an antisymmetric matrix A in (13). Now if we have pressure release top and a rigid bottom, the eigenvalues and eigenfunctions are

$$\phi_p \approx \sqrt{2/_h}\sin\sqrt{\lambda_p}z\,, \quad \sqrt{\lambda_p} \approx \frac{(p-1/_2)\pi}{h}\,, \quad q_r \approx \epsilon\sin\frac{m\pi z}{h},$$

(the last is arbitrary) The inner product in a_{jp} in (8) is approximated by

$$(q_r\phi_j,\phi_p) = 2\epsilon m\pi^{-1}(-1)^{m+p+j}(c_{jp}^2 - m^2)^{-1};$$

$c_{jp} = p - j$ when $j+p+m$ is odd, and $c_{jp} = p + j - 1$ when $j+p+m$ is even. Note that the interaction couples a wide range of frequencies, although when various sums and differences of j, p, and m are small, there is greater resonance.

(B) If $(b_j)_{j=1,2,\ldots}$ decay fast enough, repeated use of (11) gives

$$(P[\partial_z^2+q]u,\phi_p) = P(-\lambda_p)\cdot(u,\phi_p) \qquad (16)$$

for polynomial $P(\lambda)$ Convergence arguments for (16) give that a function f may be rewritten in matrix form $f[\partial_z^2+q]u \to f(-\Lambda)B$, where

$$f(-\Lambda) = \mathrm{diag}(f(-\lambda_1)\,,\ f(-\lambda_2)\,,\ f(-\lambda_3)\,,\ \ldots) \qquad (17)$$

A particularly important use of this relation occurs with the factorization of $[k^2\partial_r^2 + \partial_z^2 + q + k^2]u = 0$ into the general parabolic operators

$$(ik\partial_r - Q)(ik\partial_r + Q) - ik[\partial_r,Q]u = 0 \qquad (18)$$

where $Q = \sqrt{\partial_z^2+q+k^2}$ and $[M,N] = MN - NM$ is the commutator. Q transforms to the diagonal matrix operator $C = \sqrt{k^2-\Lambda}$ as described by (17) This permits the exact calculation of the corresponding commutator $[\frac{d}{dr} + A, C] = G$ where

$$G_{mp} = -(q_r\phi_m,\phi_p)\left(\sqrt{k^2-\lambda_m} + \sqrt{k^2-\lambda_p}\right)^{-1}.$$

G exists as long as $-\lambda_p \neq k^2$. Thus, the approximation may breakdown when $\lambda_p \approx k^2$, which may be estimated by (4) In fact if we have continuous spectrum in this range, there will necessarily be a breakdown in the commutator, and possibly in the approximation of the elliptic equation by the parabolic.

(C) Now use $\sqrt{\partial_z^2+q+k^2} \to \sqrt{k^2-\Lambda}$ to convert

$$\left(\partial_r + \frac{1}{ik}\sqrt{\partial_z^2+q+k^2}\right)u = 0 \qquad (19)$$

to the matrix differential equation

$$B_{,r} = VB\,, \quad V \equiv \left(-A - \frac{1}{ik}\sqrt{k^2-\Lambda}\right). \qquad (20)$$

Pick the sign of the radical in (20) to be (a) positive when $k^2 > \lambda_j$ and to be (b) positive imaginary when $k^2 < \lambda_j$. Let P_p be the matrix with ones on the diagonal in case (a), holds and zeros otherwise, and P_e have ones on the diagonal in case (b) and zeros otherwise. Because of the choice of sign, P_p and P_e are projections onto the propagating modes and the evanescent modes, respectively. $P_p + P_e = I$, the identity matrix. Write $B_p = P_pB = (b_1,\ldots,b_m)^T$, the propagating modes of B. For these modes we may approximate the radical

$$(ik)^{-1}\sqrt{k^2-\Lambda} \approx -i - (2ik^2)^{-1}\Lambda$$

fairly well, since the maximum error is $\frac{1}{2}$. This approximation gives (13) when we include a complex exponential $\tilde{u} = ue^{ikr}$. In case (b), however, this approximation fails, since one may show $(B_e^*B_e)_{,r} \leq 0$ Thus u_e decays with respect to r, as opposed to energy conservation in (15) Padé approximants of the square root may also cause problems, in that, if one of the λ_j is near a pole of the Padé approximant, it could induce rapid oscillations not due to the original operator. One way around this is to use separate approximations for the propagating and evanescent parts. A Padé approximation should work well for the evanescent modes[2].

(D) V in (20) may be viewed as a generator of a contraction semigroup of operators, if it includes some constant attenuation. If q is a Markov process of r, Hersh and Papanicolaou [9] discuss the existence of an averaged operator $\langle V\rangle$. This could be used to estimate an average parabolic equation for (19).

4 SUMMARY

The representation of the parabolic equation in range dependent normal modes allows us to write an infinite dimensional matrix equation From this we see that mode coupling is due to mode interaction with the range dependence of the index of refraction. The generalized parabolic equation (19) gives fairly good approximation of the elliptic wave equation, except in the spectral range where propagating modes go over to evanescent modes When one approximates Q by simpler operators, one should approximate it separately on the propagating and evanescent parts of the spectrum, and then introduce the coupling between these parts.

ACKNOWLEDGEMENTS - This work was supported by the Office of Naval Research and NOARL under PE 06011153N, Dr Melbourne Briscoe, Program Manager. NOARL contribution No PR91 044-221 Approved for public release, distribution is unlimited.

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